Simple Derivation of the Thermal Noise Formula Using Window-Limited Fourier Transforms and Other Conundrums

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Abstract— **A simple theoretical derivation for obtaining the Johnson thermal noise formula using window-limited Fourier transforms is presented in detail for the first time, utilizing the well-known energy theorems. In the literature, a diverse range of alternative methods already exist, and the pedagogical value of the Fourier transform approach illustrates useful mathematical principles, taught at the undergraduate level, naturally highlighting a number of physical assumptions that are not always clearly dealt with. We also proceed to survey a number of misconceptions, problems, surprises, and conundrums concerning thermal noise.**

I. A Brief Early History

THERMAL noise caused by electrons jostled within a conductor's lattice is an electrical analogy of Brownian motion. The random motion of particles in a fluid is named after R. Brown in recognition of his work in 1827 [1]. Inspired by discoveries following a historic voyage to Australia [2], he was led to closely examine the structure of pollen under a microscope, whereupon he became intrigued by their random motion in a fluid. Brown was not the first to see such motion, in fact, many, such as W. F. Gleichen, J. T. Needham, G.-L. Leclerc, A.-T. Brogniart, and L. Spallanzani [1], [3], had seen it before except that their ability for correct interpretation was clouded amidst the ongoing debate on vitalism and spontaneous generation. Brown opened the door for research in microscopic fluctuations by being the first to perform a major systematic experimental analysis convincingly demonstrating that the motion was not due to bubbles, release of matter, interaction between the particles themselves or organisms. It is interesting to note, however, that J. Ingen-Housz in 1784 [4] and then J. Bywater in 1819 [5] independently came to the conclusion—before Brown—that the motion exists for inorganic particles. For the next half a century a number of scientists, including H. V. Regnault, L. C. Wiener [7], Cantoni and Oehl, and S. Enxer [6] debated whether it was heat, light or electrical forces causing the fluctuations. Finally, in 1877, R. J. Delsaulx for the first time suggested impact of liquid molecules on the particles [8]. Evidence to support this hypothesis came gradually. The work of L-G. Gouy [9] showed the motion slowed down in more viscous fluids, and he took the further step of ascribing fluctuations to *thermal* motion of the fluid molecules. F. M. Exner, in 1900, established that the activity decreased with increasing particle size and decreasing temperature [10]. About this time, M. R. Smoluchowski began theoretical work on the subject and published papers in the 1904–1906 period. A. Einstein independently wrote a number of famous theoretical papers in 1905–1907. The first theoretical discussion of electrons as Brownian particles [11] came as early as 1912, by G. L. de Haas-Lorentz,[1] which inspired G. A. Ising, in 1925, to fully explain the problem of galvanometer fluctuations [13] observed by Moll and Burger [14]. With J. J. Thomson's discovery of the electron in 1897 and P. K. L. Drude's classical model of electrical conduction in terms of an electron gas in an atomic lattice, both well established by this stage, the accumulated knowledge was ripe for the understanding of electrical noise.

J. B. Johnson (Fig. 1), drawing inspiration from W. Schottky's work [16] of 1918, began in 1925 to characterize the thermal noise in various conductors via a vacuum tube amplifier and published in 1927–28 his well-known formula [15] for voltage noise, which is equivalent to Einstein's fluctuation formula for Brownian motion of charge. Johnson discussed his results with H. Nyquist (Fig. 2) who, about a month later, managed to produce a remarkably compact theoretical derivation based on the thermodynamics of a transmission line [17].

Because of the equivalence of Johnson's formula with the earlier theory (see also [18]), some authors prefer to use the neutral term *thermal noise*, whereas as some prefer *Johnson noise* or *Johnson-Nyquist noise* to prevent the confusion between electrical thermal noise and *temperature fluctuations*. Similarly, W. S. Jevons in 1878 attempted to coin the phrase *pedesis* (Gk. 'jump') [20] as a neutral expression for Brownian motion; however, tradition prevailed. For a brief chronology see Table I.

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[1] She was the eldest daughter of the physicist H. A. Lorentz who married his assistant W. J. de Haas and has the distinction of being the first woman in noise theory. In his 1912 series of lectures, H. A. Lorentz expounded her work within a statistical thermodynamics framework [12].

Fig. 1. John [Erik] Bertrand Johnson (*né* Johan Erik Bertrand) was born in the Carl Johan parish of Goteborg, Sweden, on October 2nd, 1887, and christened October 7th, 1887. His birth certificate only records his mother Augusta Mathilda Johansdotter (b. 1866) and his surname derives from his assumed father Carl Bertrand Johansson. He emigrated to the United States in 1904 and attended Yale University at the same time as Nyquist, producing a thesis entitled "Total Ionisation of Slow Electrons" in 1917. Johnson was a pioneer in the study of cathode ray tubes and studied the causes of noise in vacuum tubes in the 1925–1930 period, working at Bell Telephone Laboratories until 1952. He then joined the Edison Research Laboratory until his retirement in 1969. He received a number of awards and medals and held over 30 patents. A Republican and Presbyterian, Johnson became a US citizen in 1938. His interests included opera, plant life, and woodwork. He married Clara Louisa Conger (d. 1961) in 1919 and Ruth Marie Severtson Bowden in 1961. By his first marriage he had two sons, Bertrand Conger and Alan William. Bert Johnson died at the age of 83 in Orange, NJ, on November 27, 1970.

Fig. 2. Harry Nyquist (*né* Harry Theodor Nyqvist) was born in the parish of Stora Kil in the county of Varmland, Sweden, February 7, 1889, the son of Lars Jonsson Nyqvist (b. 1847) and Katrina Eriksdotter (b. 1857). There were seven children altogether: Elin Teresia, Astrid, Selma, Harry Theodor, Amelie, Olga Maria, and Axel, none of whom were christened. Harry emigrated to the United States in 1907 and attended Yale University. His 1917 thesis was on the Stark effect and, therefore, he would have been aware of the work of H. A. Lorentz; however, no historian has yet established if Nyquist knew of Lorentz's 1912 work [12] on the statistical thermodynamics of noise. Nyquist began working with the AT&T Company in 1917 and went on to produce 138 patents in the area of telephone and television transmission, as well as collecting many honors and awards. He arrived at his derivation of the thermal noise formula about a month after discussions with Johnson. He is also credited with the Nyquist diagram for defining stable conditions in negative feedback systems and the Nyquist sampling theory in digital communications. Harry Nyquist was unique in that he was famous as a theoretician and yet was a prolific inventor. He retired in 1954, although he continued as a consultant, and died at the age of 87 on April 4, 1976, in Harlingen, TX.

II. Overview of Methods in the Literature

The three most common methods found in the pedagogical literature for the derivation of Johnson's formula are 1) Nyquist's original proof [17] considering a transmission line in thermal equilibrium [21]–[27], 2) a sharply tuned LCR circuit in thermal equilibrium [28], and 3) the autocorrelation function technique [26], [29], [30].

Other techniques concentrate on starting from individual particle motion include the 1) Langevin equation approach considering particle mobilities and possible use of the Wiener-Khintchine theorem [21], [22], 2) kinetic theory derivation using the simple Drude model picture of conduction in metals in terms of a classical electron gas [23], [31], 3) extension of this approach considering the modern Fermi-Dirac gas model of electron conduction [32], and 4) a further generalized statistical proof independent of whether particles are classical or quantum [24].

This remarkable diversity of proofs allows the pedagogue to draw upon whichever suits the particular course material at hand. However, none of the cited references present a proof in terms of Fourier transforms, making use of the well-known *energy theorems*. The notion that the energy in a stationary random process is infinite is partially responsible for this omission. As pointed out in [21] and [25], the use of Fourier transforms is nevertheless permissible as the *power* is finite, however they do not pursue the matter any further. Therefore, for the first time, we shall detail a proof using the Fourier transform energy theorems by considering them in terms of power.

Nyquist's original derivation has been criticized as it only considers TEM modes and part of the proof involves shorting out the resistors, leaving an unanswered question of upset thermal equilibrium. The proof can be modified to overcome such objections [18], [26], at the expense of brevity. A further pedagogical objection is that the Nyquist proof and the tuned LCR proof explicitly say very little about the statistical assumptions of the noise process; a list of further objections is given by [19]. The present alternatives are either lengthy Lévy-Khintchine-Paley-Wiener type formalisms or kinetic theory. Thus, our aim is for a simple "engineering proof" based on Fourier transforms.

TABLE I
HISTORY OF FLUCTUATION RESEARCH—A BRIEF EARLY CHRONOLOGY

Name	Background	b.-d.	Origin	Observation	Date
Sacharias Jansen	Optician, Coin forgery	1588-c.1631	Dutch	Invented compound microscope with father	≤1609
Hans Lipperhey	Spectacle maker	c.1570-1619	Dutch	Independently invented compound microscope	≤1609
Antoni van Leeuwenhoek	Anatomy, Microscopy	1632-1723	Dutch	Extensive microscope observations	-
William Derham	Bishop, Physician	1657-1735	English	Observed animacules in pepper water	1713
Georges Louis Leclerc (*Comte de Buffon*)	Naturalist	1707-1788	French	Observed the motion before Brown	-
John Turberville Needham	Naturalist, Clergyman	1713-1781	Eng-Bel	Observed the motion before Brown	-
Wilhelm F. von Gleichen-Russworm	Naturalist	1717-1783	German	Observed the motion before Brown	-
Lazzaro Spallanzani	Naturalist, Jesuit	1729-1799	Italian	Observed the motion before Brown	-
Jan Ingen-Housz	Physics, Medicine	1730-1799	Dut-Eng	Observed inorganic particle motion	1784
John Bywater	Optician, Philos.	c.1774-1836	English	Observed inorganic particle motion	1819
Robert Brown	Botany	1773-1858	Scottish	First systematic study of the motion	1827
Henri Victor Regnault	Physics, Chemistry	1810-1878	French	Thought the motion was due to light	1858
(Ludwig) Christian Wiener	Math., Physics, Philos.	1826-1896	German	Discarded evaporation as an explanation	1863
Cantoni & Oehl	Physics		Italian	Found motion persists after a year	1865
Sigmund Exner	Medicine, Physiol.	1846-1926	German	Found smaller particles move quicker	1867
(Rene) Joseph Delsaulx	Math., Physics, Priest	1828-1891	Belgian	First to suggest molecular impact	1877
William Stanley Jevons	Logic, Economics	1835-1882	English	Tried to coin the term *pedesis*	1878
Carl Wilhelm von Nageli	Botany, Microscopy	1817-1891	Swiss	Incorrectly discards molecular impact idea	1879
William Miller Ord	Anatomy	1843-1902	English	Argues against electrical cause	1879
Louis-Georges Gouy	General Physics, Optics	1854-1926	French	*Motion more rapid if viscosity lowered*	1888
(Richard) Meade Bache	Physics	c.1830-1907	USA	Motion persists after a week in darkness	1894
Joseph John Thompson	Physics	1856-1940	English	Discovered the electron	1897
Paul Karl Ludwig Drude	Physics	1863-1906	German	Electron gas model of conduction	1900
Felix Maria Exner	Meteorology	1876-1930	German	Motion increases with temperature	1900
Louis Jean Baptiste Alphonse Bachelier	Mathematics	1870-1946	French	Analyzed fluctuations in Paris stock exchange first to apply theory to Brownian motion	1900
Jean Baptiste Perrin	Physics	1870-1942	French	Began systematic experiments.	1900
Marian Ritter von Smolan Smoluchowski	Physics	1872-1917	Polish	First systematic theory began	1900
Albert Einstein	Physics	1879-1955	Ger-USA	Began publishing famous theoretical papers	1905
Geertruida Luberta de Haas-Lorentz	Physics	1885-1973	Dutch	First to discuss electrical noise and first woman in noise theory	1912
Hendrik Antoon Lorentz	Physics	1853-1928	Dutch	Statistical thermodynamics framework	1912
Walter Schottky	Physics	1886-1976	German	Classic paper on electrical noise	1918
Moll & Burger	Physics		Dutch	Amplified galvanometer fluctuations	1925
Gustav Adolf Ising	Physics	1883-1960	Swedish	Correctly explained galvanometer noise	1926
(John) Bert(rand) Johnson	Physical electronics	1887-1970	Swd-USA	Began work on circuit noise	1925
Harry (Theodor) Nyquist	Comms. Engineering	1889-1976	Swd-USA	Transmission line based derivation	1927
Norbert Wiener	Mathematics	1894-1964	USA	Began mathematical formalism	1928

Section III introduces the lumped circuit model, Section IV discusses windowed Fourier transform concepts, Section V derives Johnson's formula, and subsequent sections review a number of the conundrums, debates and, anomalies surrounding thermal noise that are generally not clearly discussed in the literature.

III. THE LUMPED MODEL

Consider a resistor in parallel with a capacitor. Any segment, dx, of this circuit loop consists of some continuous conducting medium that has some finite resistance, e.g., the resistor material or the metal wires or the capacitor dielectric.

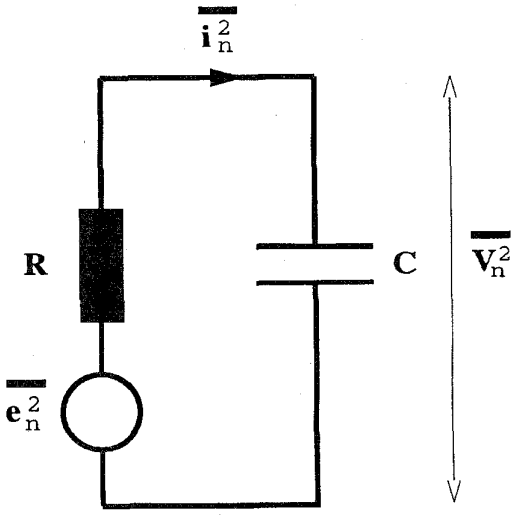

Fig. 3. Lumped circuit model.

The electrons in these materials will experience random velocity fluctuations, due to thermal energy in the material. This Brownian-like motion of charge, leads to a voltage fluctuation across each segment of the circuit. In a given instant of time, the sum of this ensemble of fluctuations forms a net voltage e_n.

Assumption 1: This circuit is modeled in Fig. 3 by a random voltage generator $e_n(t)$, a pure capacitor C, a lumped resistor R, and resistance-free wires. Note that the capacitor is pure, so there is space between the plates, and we therefore expect the thermal noise formula to be independent of C.

IV. The Windowed Fourier Transform

By definition the Fourier transform $I_n(\omega)$ of the fluctuating current $i_n(t)$ through the circuit, as shown in Fig. 3, is given by

$$I_n(\omega) = \int_{-\infty}^{\infty} i_n(t) e^{-j\omega t}\, dt. \tag{1}$$

The Fourier transform is a useful tool for this noise problem, as undergraduate level students are widely taught convenient *energy theorems*—including how to express voltage power spectral density in terms of the transform. So, the approach that we present here is to use the transforms to find an expression for the power spectrum in the noise generator in terms the power spectrum in the capacitor. This expression can then be reduced in terms of mean squared voltage fluctuations, followed by the standard equipartition theory arguments to finally yield the thermal noise formula.

However, $I_n(\omega)$ can only exist if $i_n(t)$ is absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |i_n(t)|\, dt < +\infty.$$

Unfortunately, this condition is not satisfied as $i_n(t)$ is a randomly varying function of time and does not decay to zero as $t \to \pm\infty$. The instantaneous values of $i_n(t)$ cannot be predicted and this type of function represents an example of a *stochastic process*.

A common student error is to ignore the integrability problem from the outset and to proceed to express $I_n(\omega)$ in terms of $V_n(\omega)$, the transform of $v_n(t)$, by observing that

$$I_n(\omega) = \int_{-\infty}^{\infty} \frac{d(Cv_n)}{dt} e^{-j\omega t}, dt$$

and then integrating by parts

$$I_n(\omega) = C[v_n e^{-j\omega t}]_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} (-j\omega C) v_n e^{-j\omega t}\, dt$$

which yields the "correct" result

$$I_n(\omega) = j\omega C V_n(\omega)$$

by making a second mistake in assuming that $V_n(\omega)$ exists and the square bracket residual vanishes at the limits, which is clearly unfounded as the limiting values are unknown. At this point disapproval can be expressed, and the concept of windowing the function can be introduced. In practice, the random process can only be observed for a finite window of time τ, so a dimensionless time window[2] function $W(t,\tau)$ is defined in Fig. 4 and, provided a large τ is chosen to minimize statistical sampling error, the windowed version of (1) becomes

$$I_n(\omega,\tau) = \int_{-\infty}^{\infty} i_n(t,\tau) e^{-j\omega t}\, dt. \tag{2}$$

As the measured voltage v_n is only known over the observation time τ, it is tempting for the student to define the windowed current as

$$i_n(t,\tau) = \frac{d}{dt}[Cv_n(t)W(t,\tau)].$$

Assumption 2: The stochastic process is independent of where the origin of $W(t,\tau)$ is placed on the time axis. This condition is referred to as stationarity and is reasonable in view of the observed nature of thermal noise.

This useful property means that (2) is invariant to the positioning of $W(t,\tau)$. Although the presence of W in (2) solves the integrability problem, it introduces the artifact of spectrum leakage. The leakage occurs essentially because the transform of W, itself, contains a continuous range of nonzero frequency components. This can be ignored as we will eventually be considering just the limiting case as $\tau \to \infty$. Another potential problem is that the differential term appears to create a discontinuity artifact at the edges of the window, however by substitution into (2) the student finds that it eventually cancels, as follows

$$I_n(\omega,\tau) = \int_{-\infty}^{\infty} \frac{d}{dt}[Cv_n W] e^{-j\omega t}\, dt \tag{3}$$

[2] Some texts, such as [21], replace a random variable $x(t)$ by a "gated" random variable $x_T(t)$, which is zero outside the time window T. A Fourier transform is then performed on $x_T(t)$. Although, as we show, this does eventually lead to the correct result—the *implicit* window is introduced without discussion and the student is justified in questioning what becomes of any edge discontinuity effects. For this reason we introduce an *explicit* window function $W(t,\tau)$ as a "book keeping" device to track and monitor the artifact.

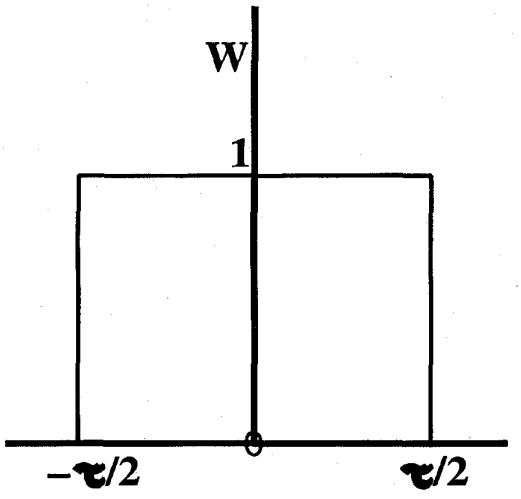

Fig. 4. The window function.

by expanding the differential we obtain

$$I_n(\omega,\tau) = C\int_{-\infty}^{\infty} \frac{dv_n}{dt} W e^{-j\omega t}\,dt + C\int_{-\infty}^{\infty} \frac{dW}{dt} v_n e^{-j\omega t}\,dt$$

integrating the left-hand integral by parts

$$\begin{aligned} I_n(\omega,\tau) = {} & C[v_n W e^{-j\omega t}]_{-\infty}^{+\infty} \\ & - C\int_{-\infty}^{\infty} v_n \left\{ \frac{dW}{dt} e^{-j\omega t} - j\omega W e^{-j\omega t} \right\} dt \\ & + C\int_{-\infty}^{\infty} \frac{dW}{dt} v_n e^{-j\omega t}\,dt \end{aligned}$$

where the residual now legitimately vanishes and the superfluous differential window terms cancel, again giving the "correct result"

$$I_n(\omega,\tau) = j\omega C V_n(\omega,\tau).$$

Although this attempt has greater merit, as the integrals are now legitimate, it can be argued that the invoked definition of windowed current was somewhat *ad hoc*. Notice as $\tau \to \infty, i_n(t,\tau)$ does not tend to $i_n(t)$ due to delta functions at the extremities. The artificial introduction of these spikes, from the outset, fortuitously cancels with the differential window term during the integration by parts, thereby producing the "correct" result. Again disapproval must be expressed and the preferred method now follows.

Consider a windowed version of $e_n(t)$ as

$$e_n(t,\tau) = e_n(t) W(t,\tau).$$

We now define the signals $v_n(t,\tau)$ and $i_n(t,\tau)$ as the *responses* of the circuit to $e_n(t,\tau)$. Notice this time that as $\tau \to \infty, v_n(t,\tau) \to v_n(t)$ and $i_n(t,\tau) \to i_n(t)$. We have $i_n(t,\tau) = \frac{d}{dt}[C v_n(t,\tau)]$, thus

$$I_n(\omega,\tau) = \int_{-\infty}^{\infty} \frac{d}{dt}[C v_n(t,\tau)] e^{-j\omega t}\,dt$$

and integrating by parts

$$I_n(\omega,\tau) = C[v_n(t,\tau) e^{-j\omega t}]_{-\infty}^{+\infty} + j\omega C\int_{-\infty}^{\infty} v_n(t,\tau) e^{-j\omega t}\,dt$$

which reduces to

$$I_n(\omega,\tau) = j\omega C V_n(\omega,\tau). \tag{4}$$

Although this result is trivial, it was important to show that there were no window artifact problems. General discussion of window problems can be found in [33] and [34]. As $\frac{d^m W}{dt^m} = 0|_{t=\pm\infty}$ it can be shown for the general case that

$$\frac{d^m v_n(t,\tau)}{dt^m} \leftrightarrow (j\omega)^m V_n(\omega,\tau)$$

is a windowed or time-limited Fourier pair.

V. The Thermal Noise Formula

We now proceed to use the (4) result to find the power spectrum in the capacitor and then produce the celebrated Johnson formula. Consider the voltages around the loop, in Fig. 3, by Kirchhoff

$$v_n(t) + i_n(t)R = e_n(t)$$

and viewed from the window $W(t,\tau)$ this becomes

$$v_n(t,\tau) + i_n(t,\tau)R = e_n(t,\tau).$$

Notice that by definition, $e_n(t,\tau)$ does not contain any delta function terms and therefore $i_n(t,\tau)$ and $v_n(t,\tau)$ must also be free of spikes. This can simply be demonstrated by *reductio ad absurdum*: if $i_n(t,\tau)$ contained a delta function pair, due to windowing, then $v_n(t,\tau)$ would need an identical pair of opposing sign to balance the above equation. This would be impossible, however, as $i_n(t,\tau)$ would then contain the second derivative and the reasoning continues inductively *ad infinitum*.

Now, taking Fourier transforms we have

$$V_n(\omega,\tau) + I_n(\omega,\tau)R = E_n(\omega,\tau).$$

substituting in (4) gives

$$V_n = \frac{E_n}{1 + j\omega RC} \tag{5}$$

multiplying by complex conjugates

$$|V_n|^2 = \frac{|E_n|^2}{1 + (\omega RC)^2}. \tag{6}$$

By conservation of energy, the total energy in the time domain must equal that in the frequency domain, therefore

$$\int_{-\infty}^{\infty} e_n^2(t,\tau)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |E_n(\omega,\tau)|^2\,d\omega.$$

This is known as *the energy theorem* or Plancherel's theorem (a special case of Parseval's theorem). Each side of the equation represents *total energy* and therefore $|E_n|^2$ represents the *energy density* with units of $V^2 s/\text{Hz}$. Due to the Hermitian

property of the Fourier transform, $|E_n|^2$ is always even therefore, we can write the one-sided form

$$\int_{-\infty}^{\infty} e_n^2(t,\tau)\,dt = \frac{1}{2\pi}\int_0^{\infty} 2|E_n(t,\tau)|^2\,d\omega.$$

By definition of temporal average

$$\begin{aligned}\langle e_n^2\rangle &= \lim_{\tau\to\infty}\langle e_n^2\rangle_\tau \\ &= \lim_{\tau\to\infty}\frac{1}{\tau}\int_{-\infty}^{\infty} e_n^2(t,\tau)\,dt\end{aligned}$$

therefore

$$\langle e_n^2\rangle = \lim_{\tau\to\infty}\frac{1}{2\pi}\int_0^{\infty} 2\frac{|E_n(\omega,\tau)|^2}{\tau}\,d\omega$$

where $\frac{|E_n|^2}{\tau}$ is called the *sample spectrum* or *periodogram*. It is permissible to move the $\lim_{\tau\to\infty}$ inside the integral provided the ensemble average is performed first [21, p. 32]. Thus

$$\overline{e_n^2} = \frac{1}{2\pi}\int_0^{\infty}\left\{\lim_{\tau\to\infty}\frac{2\overline{|E_n|^2}}{\tau}\right\}d\omega.$$

As we have a random process, the limit would be indeterminate had the ensemble average not been performed first.

Assumption 3: The process is ergodic, so temporal and ensemble averages are equivalent, i.e., $\lim_{\tau\to\infty}\langle e_n^2\rangle_\tau = \overline{e_n^2}$. Thus

$$\langle e_n^2\rangle = \frac{1}{2\pi}\int_0^{\infty}\left\{\lim_{\tau\to\infty}\frac{2\overline{|E_n|^2}}{\tau}\right\}d\omega. \tag{7}$$

By definition, the one-sided power spectral density of e_n is $S(\omega) = \lim_{\tau\to\infty}\frac{2\overline{|E_n|^2}}{\tau}$, therefore we can rewrite (7) as

$$\langle e_n^2\rangle = \frac{1}{2\pi}\int_0^{\infty} S(\omega)\,d\omega. \tag{8}$$

Assumption 4: The noise spectrum is white, therefore $S(\omega) = S_0$, a constant.

For a practical measuring instrument bandwidth of $\Delta\omega$, (8) becomes

$$\langle e_n^2\rangle = \frac{1}{2\pi}S_0\Delta\omega. \tag{9}$$

Using identical arguments for the capacitor voltage, v_n, we have

$$\langle v_n^2\rangle = \frac{1}{2\pi}\int_0^{\infty}\left\{\lim_{\tau\to\infty}\frac{2|V_n|^2}{\tau}\right\}d\omega$$

and substituting in (6)

$$\begin{aligned}\langle v_n^2\rangle &= \frac{1}{2\pi}\int_0^{\infty}\frac{1}{1+(\omega RC)^2}\left\{\lim_{\tau\to\infty}\frac{2|E_n|^2}{\tau}\right\}d\omega \\ &= \frac{S_0}{2\pi}\int_0^{\infty}\frac{d\omega}{1+(\omega RC)^2} \\ &= \frac{1}{2\pi}\frac{S_0}{RC}[\arctan(\omega RC)]_0^{\infty} = \frac{1}{2\pi}S_0\frac{\pi}{2RC}.\end{aligned}$$

Putting this into (9), to eliminate S_0, gives

$$\langle e_n^2\rangle = \frac{2}{\pi}C\langle v_n^2\rangle R\Delta\omega. \tag{10}$$

Assumption 5: The system is in equilibrium with its surroundings.

According to equipartition theory, a general dynamical system in thermal equilibrium has on average a potential energy of $kT/2$ for each degree of freedom. One short hand method for counting up the degrees of freedom in a linear system is to count the number of independent quadratic variables in the energy expression. By inspection of (10), we see that our system takes up energy as $\frac{1}{2}C\langle v_n^2\rangle$ and, therefore, it has one degree of freedom, hence

$$\frac{1}{2}C\langle v_n^2\rangle = \frac{1}{2}kT. \tag{11}$$

Assumption 6: Let us assume the system is classical (i.e., no quantum effects) so that the Maxwell-Boltzmann kT term holds.

Substituting (11) into (10) finally yields Johnson's formula for open circuit noise voltage

$$\langle e_n^2\rangle = \frac{2}{\pi}kTR\Delta\omega \tag{12}$$

$$= 4kTR\Delta f. \tag{13}$$

This unassuming equation is a source of a number of interesting conundrums and much student consternation. One common question is "where does the coefficient of four *really* come from?" That is, "what is its physical significance?" This can now be quite easily traced from the above analysis, where the $\frac{2}{\pi}$ clearly comes the one-sided integral of the arctan function. Hence, it is purely a "geometrical" quantity. If the integral is modified by substituting the capacitor with a more complex network, the number of degrees of freedom of the system changes to always maintain the ubiquitous four.

If the capacitor is replaced by an inductor L, the analysis can be repeated in the current domain and the generated short circuit noise current can be shown to be $\langle i_{sc}^2\rangle = \frac{2}{\pi}\frac{L}{R}\langle i_n^2\rangle\Delta\omega$, where $\langle i_n^2\rangle$ is the observed noise current. The system now takes up energy as $\frac{1}{2}L\langle i_n^2\rangle = \frac{1}{2}kT$, therefore

$$\langle i_{sc}^2\rangle = \frac{4kT\Delta f}{R} \tag{14}$$

which is the familiar current form for the Johnson noise formula.

Complete analysis and detailed discussion of the behavior of (13) and (14) at limiting values of the main variables is lacking in the pedagogical literature, so we proceed for the first time to examine, in detail, the main problem areas in the following sections.

VI. The Classical Energy Catastrophe

The most obvious problem with (13) is that it classically predicts infinite energy as $f \to \infty$. This is analogous to the black-body radiation problem where the Rayleigh-Jean's law suffers from the so-called ultraviolet catastrophe—the divergent curve having infinite area over all frequencies. Anticipating this, Nyquist [17] in 1928 suggested replacing kT with the one-dimensional form of Planck's law

$$\frac{hf}{e^{hf/kT}-1}$$

which reduces to kT as $f \to 0$ and rolls off for $hf > kT$. So far so good, however, this quantum term predicts zero energy at $T = 0$, which is a violation of the Uncertainty Principle. As we shall see the solution to this creates a further conundrum.

VII. The Quantum Energy Catastrophe

During 1911–1912, Planck's "second theory" produced the following modification to the quantum term [35]

$$\frac{hf}{e^{hf/kT} - 1} + \frac{hf}{2} = hf \coth\left(\frac{hf}{2kT}\right).$$

The extra $hf/2$ term is called the *zero-point energy* (ZPE) and in this case, at $T = 0$, the Uncertainty Principle is not violated. This creates a further conundrum in that $hf/2$ is infinite when integrated over all frequencies, which is an apparent return to the type of "catastrophe" problem we saw in the classical case. One can only assume that Nyquist accordingly did not suggest this form and probably would have been aware of Planck's own misgivings concerning the experimental objectivity of $hf/2$. The inclusion of $hf/2$ in standard noise texts only became popular after 1951 following the classic work of Callen and Welton [36] that produced the $hf/2$ ZPE term as a natural consequence of their generalized treatment of noise in irreversible systems using perturbation theory.

The solution to the catastrophe problem is that $hf/2$, in fact, turns out to be the *ground state* of a quantum mechanical oscillator. If n is the quantum number, which is a positive integer, then the allowed energy states for a quantum oscillator are $(n + \frac{1}{2})hf$, and thus the ground state is given when $n = 0$. As there is no lower energy state than the ground state, there is no energy level transition available to release the ZPE. Therefore, it can be argued that $hf/2$ should be dropped before integration of the quantum expression. This procedure is an example of *renormalization*, which basically redefines the zero of energy. Renormalization is a significant area of quantum field theory and is usually presented in a more formal manner. The problem of renormalization is an open question in the theory of gravitation where there is the apparent catastrophe of *total* energy becoming infinite. For most laboratory measurements, there is no catastrophe as we are only interested in energy *differences*.

The fact that the ground state energy (ZPE) cannot be released means that texts that quote the Callen and Welton $hf/2$ term as an observable noise component are not strictly correct. However, by coincidence it turns out that the mean square of the zero point fluctuation (ZPF) also has the form $hf/2$ [37]. The mean square does not vanish with renormalization, of course, and this ensures the Uncertainty Principle survives renormalization. The mean square fluctuation is a detectable quantity and represents the magnitude of the ZPF.

Each mode contributes $hf/2$ toward the mean square fluctuation and, for an infinite number of frequencies, the magnitude is infinite. It is considered that this infinity is not fundamental, since the measurement conditions have not been specified. It can be shown [37] that for any finite observation bandwidth and volume of space the magnitude of the fluctuations of a quantum field is finite—if either the bandwidth is infinite or the measurement is evaluated at a *point* in space then the fluctuations become infinite.

VIII. The Steak Grilling Debate

In 1982, Grau and Kleen expressed the view that $hf/2$ is both unextractable and unobservable, adding their memorable rejoinder in the *Solid-State Electronics* journal that $hf/2$ is not "available for grilling steaks" [38]. Uncannily, about the same time Koch, Van Harlingen, and Clarke (KVC) published noise measurements in superconductors reporting to have observed ZPF [39]. Over the next three to four years a number of independent superconductor papers followed, all nonchalantly quoting the KVC interpretation of ZPF as standard. In reply, Kleen (1987) essentially restated his case pointing out an unanswered question in the superconductor measurements [41]. As far as we are aware there has been no published KVC reply.

The orthodox position is that the effects of ZPF, such as in the Casimir effect [41], are observable. ZPF also has an orthodox status in explaining the observations of Mullikan [42], Lamb [43], and the nature of liquid helium [44]. On the other hand, consensus is not total as the school of Kleen has some support e.g., [26, p. 173] and [45], the commonly supposed link between spontaneous emission and ZPF has been criticized [46] and the overall understanding of ZPF is also questioned, as expressed, for example, in the following quote [47].

> "The obvious question, then, is whether the zero-point energy and the vacuum fluctuations are one and the same thing. If they are, why is it that the former can be eliminated from the theory? The answer is not yet clear, and a deeper significance has yet to be discovered. Therefore, we will adopt the view that the zero-point energies are to be formally removed from the theory..., and all physical effects of the type... discussed are to be ascribed to quantum fluctuations of the vacuum.... It must be admitted that the vacuum is not completely understood, neither physically nor philosophically. Whether or not the vacuum fluctuations are intimately related to the (unobservable) zero-point energy remains an open question."

where the expression "vacuum fluctuations" is an alternative term for ZPF. The view that ZPF cannot give rise to a detectable noise power itself, but can indirectly modulate or induce a detectable noise power has been recently expounded [48].

As for grilling steaks, the debate still sizzles but has shifted away from electrical noise theory. Controversial attempts to harness ZPE are under way using the concept of system self-organization [49] and presupposing the idea that the ground state is not the actual source of energy but a "pipeline" into some universal background source [50]. In an enterprising decade where there have been controversial attempts to consider superluminal velocity [51] and quantum information theory (promising two bits of information from one physical bit [52] and a form of teleportation [53]), there is no doubt that ZPE research will thrive. It remains to be seen what concrete

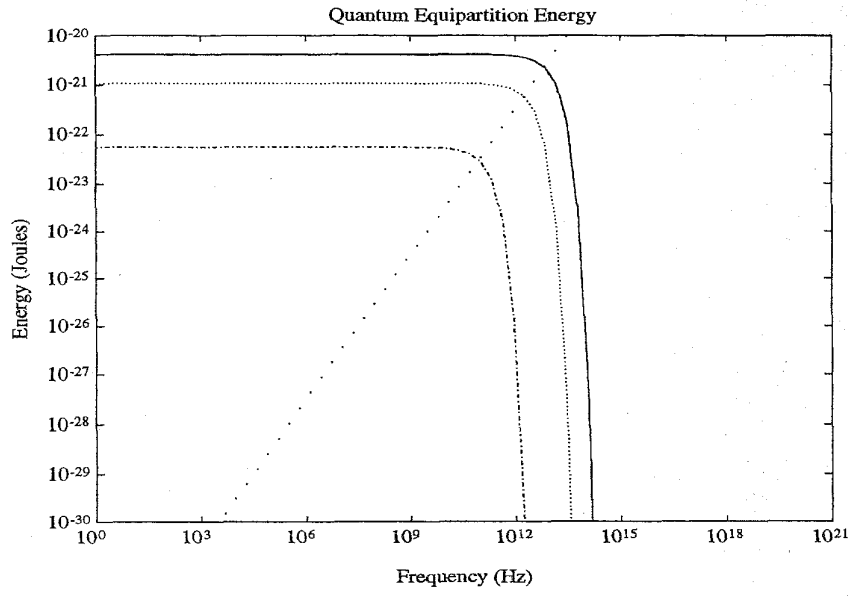

Fig. 5. Quantum equipartition energy versus frequency for 300, 77, and 4 K. The line represents the $hf/2$ term plotted separately.

results are produced and, if any, what the implications are to noise theory. Until further evidence, the quantum zero-field should be regarded as a conservative field as far as the extraction of energy is concerned.

IX. Quantum Cut-Off Experimental Status

Fig. 5 shows a theoretical plot of the quantum term for different temperatures. The $hf/2$ term is plotted to illustrate that at normal working frequencies and temperatures it is vanishingly small, so for these conditions it can be neglected regardless of the status of debate. It can be seen from the Fig. 5 plot that experimental verification of the quantum cut-off point for electrical noise is rather difficult due to the TeraHertz frequencies. If the temperature is reduced, to reduce the cut-off frequency, we see that the maximum energy of the curves falls, thus making noise detection more difficult. In 1981, van der Ziel [54] proposed to make measurements in this region at 100 GHz using Hanbury-Brown Twiss circuits; unfortunately, this research effort was never completed. The only curves we have today, for electrical noise, appear to be those of the type of KVC, which show no cut-off due to ZPF becoming significant. Therefore, as far as we are aware, there are to this day no measurements that directly demonstrate the quantum cut-off for *electrical* thermal noise. Although the cut-off region, for electrical noise, has so far been obscured by ZPF it may become possible in the future to view at least part of this region, without violation of the Uncertainty Principle, if somehow the concept of *squeezed states* can be successfully employed for the electrical case (e.g., [55]).

X. MacDonald's Objection

In 1962, MacDonald raised an interesting objection [56] concerning the quantum term. He correctly demonstrated that for $hf > kT$ the time-dependence characteristic of a *reversible* system is exhibited. For $hf \ll kT$ he showed that the system is *irreversible*, as expected. Given that an electrical resistor is regarded as a dissipative irreversible system, a transition to a reversible regime for $hf > kT$ caused MacDonald to doubt the validity of the quantum term altogether.

The transition to the reversible regime can be simply thought of as taking place because at high frequencies, $hf > kT$, i.e., for time intervals less than h/kT, the period is too short to achieve thermal equilibrium. As noise is a manifestation of a dissipative system maintaining thermal equilibrium, if the intervals are too short, then the dissipative process must roll off at these higher frequencies. This is precisely what is predicted by the quantum term.

Dissipation can be thought of as a process that eventually brings classical particles, in a closed system, to rest. This situation is not permissible for quantum particles as it would be a violation of the Uncertainty Principle. Dissipation does not play a role in microscopic description of quantum particles. It is a macroscopic concept whose relation to the quantum microscopic description is purely a statistical one in the classical limit. The "sleight of hand" that turns nondissipative equations of motion into dissipative ones in the classical limit is hidden within the equation boundary conditions. For a more mathematical discussion see [57]. From the modern viewpoint, quantum Brownian motion (QBM) is now a major discipline area [58] and MacDonald's objection is therefore clearly outmoded.

XI. The Case of Limiting R and C

Remembering that the Johnson expression for $\langle e_n^2 \rangle$ is the case for an *open circuit* resistor, we now systematically illustrate, for the first time, how to examine the output voltage, current, and charge fluctuations ($\langle v_n^2 \rangle$,$\langle i_n^2 \rangle$ and $\langle q_n^2 \rangle$) for the various limiting cases of R and C. The results are summarized in Table II.

TABLE II
THERMAL NOISE OVER INFINITE BANDWIDTH FOR DIFFERENT CASES OF LIMITING R AND C

		Classical			Quantum		
		$\langle v_n^2 \rangle$	$\langle i_n^2 \rangle$	$\langle q_n^2 \rangle$	$\langle v_n^2 \rangle$	$\langle i_n^2 \rangle$	$\langle q_n^2 \rangle$
$R \to 0$	Shorted Cap.	0	0	0	0	0	0
$R \to \infty$	Open Cap.	$\frac{kT}{C}$ (dc)	0	kTC (dc)	$\frac{kT}{C}$ (dc)	0	kTC (dc)
$C \to \infty$	Shorted Res.	0	∞	∞	0	$\frac{2}{3hR}(\pi kT)^2$	∞
$C \to 0$	Open Res.	∞	0	0	$\frac{2R}{3h}(\pi kT)^2$	0	0

A. The Case of Finite R and C and $f \to \infty$

The voltage noise detected across the capacitor is simply

$$\langle v_n^2 \rangle = \int_0^\infty \frac{4kTRdf}{1+(2\pi fRC)^2} = \frac{kT}{C}$$

which normally causes some surprise as the kT/C term is independent of R, which is the source of the noise! This is easily explained by observing that as R increases, the corresponding increase in noise is exactly canceled by a decrease in circuit bandwidth $\propto 1/R$. The circuit current noise is given by

$$\langle i_n^2 \rangle = \int_0^\infty \frac{(4kT/R)(2\pi fRC)^2 df}{1+(2\pi fRC)^2} = \infty$$

and this divergent result is a consequence of classical theory breakdown.

B. The Case of Finite f, R and C

For a finite frequency band Δf, taken from zero, the voltage noise becomes

$$\langle v_n^2 \rangle = \frac{2kT}{\pi C} \arctan(2\pi \Delta f RC) \approx \frac{kT}{C} - \frac{4kTR\Delta f}{(B_m/B_c)^2}$$

by expanding arctan for $B_m \gg B_c$, where the measurement bandwidth $B_m = (\pi/2)\Delta f$ and the circuit bandwidth $B_c = 1/(4RC)$. This remarkable result is the difference between the familiar kT/C noise and the open circuit noise divided by the square of the ratio of the bandwidths. The analogous expression for the current noise can be shown to be

$$\langle i_n^2 \rangle \approx \frac{4kT\Delta f}{R}\left(1+\left(\frac{B_c}{B_m}\right)^2\right) - \frac{kT}{CR^2}$$

which goes to infinity for $\Delta f \to \infty$, as expected for the classical theory. The solutions using the quantum term for finite R and C involve tedious integrals and are not that instructive; for such a treatment refer to [59].

C. Open Circuit Capacitor: Finite C, $R \to \infty$, $f \to \infty$

Using the well-known delta function approximation of the form

$$\delta(\alpha x) = \frac{1}{\pi} \lim_{x \to \infty} \left\{ \frac{x}{1+\alpha^2 x^2} \right\}$$

we observe that

$$\lim_{R \to \infty} \left\{ \frac{R}{1+(2\pi fRC)^2} \right\} = \pi \delta(2\pi fC) = \frac{\delta(f)}{2C}$$

by applying the delta function identity $\delta(\alpha x) = \delta(x)/\alpha$. Therefore

$$\langle v_n^2 \rangle \big|_{R=\infty} = \frac{1}{2} \int_{-\infty}^{\infty} 4kT \frac{\delta(f)}{2C} df = \frac{kT}{C}$$

where the integral limits are taken between $\pm\infty$ as $\delta(f)$ is centered about the $f = 0$ axis. The factor of $1/2$ is introduced as we are dealing with frequencies in the positive domain—this is justified as the function is symmetrical about the $f = 0$ axis. This surprising result of kT/C has to be interpreted in terms of a dc voltage across the capacitor, because as $R \to \infty$ the circuit bandwidth$\to 0$. Therefore, the classical formulation predicts that an ensemble of capacitors will, on average, display a dc voltage of $\sqrt{kT/C}$ across their terminals. The source of the dc voltage can be thought of as the voltage that is sampled by the capacitor at the moment the finite resistor is removed. Note that substitution of the quantum term into the above integral also produces the same result.

This explanation in terms of a dc voltage is much more satisfactory than that of [59], which considers it as a time varying noise voltage and, consequently, proposes highly ingenious ways of making a pure capacitor dissipative! This is clearly unnecessary as any dissipative proposal for the capacitor can be modeled by an equivalent resistor.

Multiplication by C^2 gives the well-known result $\langle q_n^2 \rangle = kTC$ used for analyzing noise on switched capacitor circuits. For example, a 1-pF capacitor will have 64 μV_{rms} across its terminals, or in terms of charge, this is 400 rms electrons.

D. Short Circuit Capacitor: Finite C, $R \to 0$, $f \to \infty$

As $R \to 0$, all the integrals, classical and quantum, trivially go to zero. This is expected as R is the source of the noise.

E. Open Circuit Resistor: Finite R, $C \to 0$, $f \to \infty$

For the classical case $\langle v_n^2 \rangle \to \infty$ as expected due to breakdown of the theory. The integrals for $\langle i_n^2 \rangle$ and $\langle q_n^2 \rangle$ trivially tend to zero. This is expected as there is no circuit loop for current to flow. The quantum case for the open circuit

resistor, where Γ is the mathematical gamma function and ζ is the Weierstrass zeta function, now becomes [60]

$$\langle v_n^2 \rangle = \int_0^\infty \frac{4Rhf\,df}{e^{hf/kT}-1} = 4Rh\left(\frac{kT}{h}\right)^2 \Gamma(2)\zeta(2) = \frac{2R}{3h}(\pi kT)^2$$

which implies that the open circuit thermal noise voltage across a 1 MΩ resistor, at room temperature, viewed with an instrument of infinite input impedance and infinite bandwidth (or at least $\gg kT/h = 6250$ GHz) with no parasitic capacitance present is 0.41 V_{rms}! This is rather large, but fortunately these ideal conditions are unrealisable in practice. It also assumes that the result is directly observable and not swamped by $hf/2$ quantum noise.

F. Short Circuit Resistor: Finite R, $C \to \infty$, $f \to \infty$

For the classical case, the voltage integral goes to zero whereas current and charge go to infinity. Zero voltage is expected as the terminals are shorted and infinite current in the loop is a breakdown of the classical theory. For the quantum case for current we have

$$\langle i_n^2 \rangle = \int_0^\infty \frac{(4hf/R)df}{e^{hf/kT}-1} = \frac{2}{3hR}(\pi kT)^2$$

which is finite as expected. However, the quantum integral for charge turns out to be divergent, giving infinite charge. Notice we now have a random walk type nonstationary process. The infinite result may be seen, not as a breakdown in the classical or quantum theory, but due to the artificial construct of theoretical infinite capacitance. An infinite capacitor can be thought of as an infinite store of charge—this never occurs in practice, which is another way of saying that there is no such thing as a perfect short circuit. Notice that in the limit as $C \to \infty$, the capacitor becomes simultaneously an infinite store of charge and a perfect short. This can be resolved by thinking of $C \to \infty$ as being modeled by an ideal voltage source.

XII. Power in a Matched Load

If a resistor R develops an open circuit noise voltage of $4kTR\Delta f$, the power delivered to an equal load resistor is

$$P = \langle i_n^2 \rangle R = \frac{\langle v_n^2 \rangle}{(2R)^2} R = \frac{\langle v_n^2 \rangle}{4R} = kT\Delta f.$$

This causes some surprise as P appears to be independent of R, which is the source! To understand this, let us consider an arbitrary load R_L, so the power now becomes

$$P = 4kTR\Delta f \frac{R_L}{(R+R_L)^2}.$$

So, we see that for small R, the noise term is small and therefore the delivered power P is small; whereas for large R the potential divider term becomes small, so the delivered power is still small. Maximum power transfer occurs when $\frac{dP}{dR} = 0$, which trivially yields $R = R_L$, hence there is balance achieved between noise generation and the potential divider effect.

This analysis can, of course, be reproduced by considering a resistor R in parallel with a current noise source of $4kT\Delta f/R$. A common student error is to mechanically proceed the analysis with the power, $P = \langle i_n^2 \rangle R_L$, as before. This, of course, leads to the wrong result. For the case of a current source we must put $P = \langle v_n^2 \rangle / R_L$, giving

$$P = \frac{\langle v_n^2 \rangle}{R_L} = \frac{\langle i_n^2 \rangle R^2 R_L}{(R+R_L)^2} = \frac{4kT\Delta f}{R}\frac{R^2 R_L}{(R+R_L)^2} = 4kTR\Delta f \frac{R_L}{(R+R_L)^2}$$

which is the same result as before.

Another curious feature of the $P = kT\Delta f$ formula is that it appears to imply that for large observation times the power transfer tends to zero, whereas for small times the power transfer increases. This is simply explained by noting that the application of the thermal noise formula presumes that both resistors are in thermal equilibrium. Hence, for long observation times we expect the net power transfer to be zero; otherwise a resistor would heat up and escape thermal equilibrium. However, for a small "snapshot" of time, as the fluctuations in each resistor are uncorrelated, there must be an instantaneous transfer of power. The momentary transfers of power back and forth between the resistors, on average, add to zero. This also explains why energy cannot be harnessed from the thermal noise in a resistor, cf. Brillouin's Rectifier Paradox [61], Penfield's Motor Paradox [62], Feynman's Rachet Paradox [63], Panse's Radiation Paradox [64], and Bogner's Microwave Isolator Paradox [65]. Analogous arguments are used by some authors [21] to assert that energy cannot be extracted from ZPF, however this has been disputed [66]. The apparent infinite power as the snapshot of time approaches zero is, of course, due to breakdown of the classical kT term.

XIII. Distributed RC

Until this point, our analysis has only considered a lumped circuit model. In a given practical case, a resistor may have some distributed parasitic capacitance and thus it is instructive to analyze the noise in a distributed RC line. From standard transmission line theory, the impedance looking into an RC line with the other end shorted is

$$Z = \sqrt{\frac{R}{j\omega C}} \tanh \sqrt{j\omega RC}.$$

The voltage noise seen across the open circuit terminals is found by simply inserting $\Re(Z)$ into the Johnson noise formula [30], [67]. Therefore

$$\langle v_n^2 \rangle = 4kT\Re(Z)\Delta f = 4kT\Delta f \frac{R}{\sqrt{4\pi fRC}} \frac{\sinh\sqrt{4\pi fRC} + \sin\sqrt{4\pi fRC}}{\cosh\sqrt{4\pi fRC} + \cos\sqrt{4\pi fRC}}$$

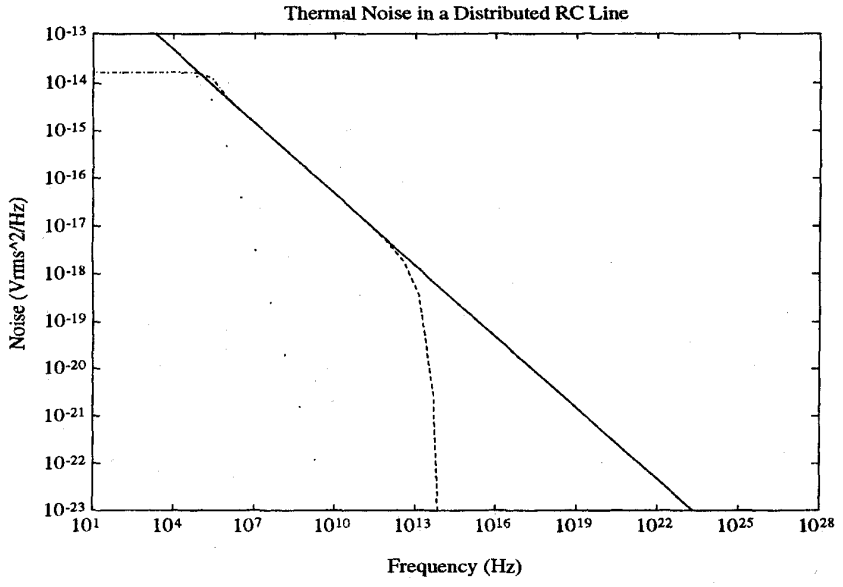

Fig. 6. Noise in distributed RC line. Chained line: short circuit load. Solid line: matched load. Dashed line: short circuit load with Planck term. $R = 1$ MΩ, $C = 1$ pF. All curves plateau at 1.66×10^{-14} V^2_{rms}/Hz, except for the matched load case that increases due to noise in the load. For reference, the simple lumped RC case is shown as a dotted line.

which reduces to $4kTR\Delta f$ for small f, as expected. This is plotted in Fig. 6 with the simple RC case for comparison, showing that at low frequencies they are equal, but at high frequencies the simple RC curve rolls off faster. The quantum curve is included to show that there is a physical limit to the slow roll-off in the distributed case. Finally, the case for a transmission line with a *matched* load is also plotted to clearly show that this option does *not* model a discrete resistor with distributed parasitic capacitance. The curious phenomenon of the noise increasing, for low frequencies, in the matched load case is a manifestation of noise in the load (which is physically unrealisable in this case) and not an anomaly in the line.

XIV. Conclusion

A brief history of the events leading up to the discovery of thermal noise has been covered, with some biographical information on Johnson and Nyquist, as these details have not been readily accessible in the pedagogical texts or encyclopedias and in some cases are misleading or incorrect. We have presented, for the first time, a simple "engineering proof" of the thermal noise formula, based on Fourier transforms, that avoids lengthy kinetic theory or Wiener formalisms and illustrates the physical assumptions more clearly than the Nyquist proof. We have also surveyed a number of debates, misconceptions, conundrums, and surprises regarding thermal noise that traditionally cause student consternation.

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